M^2 factor calculation of FLASH[1] laser beam based on *Gnuplot* 3.7® fit of the second moment propagation rule

Starting with the propagation equation of the general beam width W for any arbitrary beam, based on the second moment width definitions, we can write:

$$W_x^2(z) = W_{0x}^2 + M_x^4\left(\frac{\lambda}{\pi W_{0x}}\right)^2(z - z_{0x})^2;$$

where the general beam width is related with the second momentum of the laser beam,

$$W_x = 2\sigma_x;$$

finally obtaining:

$$\sigma_x^2(z) = \sigma_{0x}^2 + M_x^4\left(\frac{\lambda}{4\pi\sigma_{0x}}\right)^2(z - z_{0x})^2; \quad (1)$$

which is analogous for the y coordinate, just by changing y in x.

Now, since we want to fit that expression with a quadratic approximation, i.e.:

$$y(z) = a + bz + cz^2;$$

we need to arrange (1) separating the linear and the quadratic terms. By doing that, we obtain the coefficients a,b and c:

$$a = \sigma_{0x}^2 + z_{0x}^2 M_x^4\left(\frac{\lambda}{4\pi\sigma_{0x}}\right)^2; \quad (2)$$

$$b = -2z_{0x} M_x^4\left(\frac{\lambda}{4\pi\sigma_{0x}}\right)^2; \quad (3)$$

$$c = M_x^4\left(\frac{\lambda}{4\pi\sigma_{0x}}\right)^2; \quad (4)$$

Once that is done, and taking into account that we know the values of these coefficients since we are fitting the function to an array of experimental values, we just need to solve the system, being M_x^2, σ_{0x} and z_{0x} the unknown parameters.

$$z_{0x} = -\frac{1}{2}\frac{b}{c}; \quad (5)$$

then:

[1] Free electron LASer of Hamburg, at DESY.

$$a = \sigma_{0x}^2 + z_{0x}^2 c = \sigma_{0x}^2 + \frac{1}{4}\frac{b^2}{c};$$

which leads to:

$$\sigma_{0x} = \sqrt{a - \frac{b^2}{4c}}; \quad (6)$$

And finally, with this parameters calculated, we obtain the value of M_x^2 from (4):

$$M_x^2 = \sqrt{c}\frac{4\pi\sigma_{0x}}{\lambda}; \quad (7)$$

where λ is the wavelenght, which has a fixed value (in our case, 47 μm or 52 μm).

Experimental data[2]

Table 1: Experimental values of the second momentum for $\lambda = 47\ \mu m$.

z(mm)	σx(mm)	σy(mm)	z(mm)	σx(mm)	σy(mm)
2	2,3440	2,1654	19	2,2460	2,0428
3	2,3561	2,1465	20	2,2267	2,0647
4	2,4116	2,2185	21	2,0240	1,8147
5	2,3437	2,0463	22	1,9951	1,8443
6	2,3841	2,1116	23	2,1272	2,0352
7	2,3111	1,9491	24	1,9470	1,8806
8	2,3677	1,9545	25	1,9855	1,9619
9	2,4082	2,0294	26	1,8988	1,9197
10	2,3382	1,8708	27	1,9134	1,9856
11	2,3627	1,9410	28	1,8590	1,9787
12	2,3139	1,8034	29	2,1028	2,1935
13	2,2871	1,8526	30	1,8860	2,0671
14	2,3185	1,9921	35	2,0964	2,2592
15	2,1947	1,7616	40	2,1886	2,2857
16	2,1944	1,8183	45	2,3061	2,3301
17	2,1909	1,8787	50	2,3952	2,4753
18	2,1286	1,8105			

[2] The sigma values obtained had units of 'number of pixels', so I needed to multiply each value per the pixel size, which is 37 μm.

Table 2: Experimental values of the second momentum for $\lambda = 52\ \mu m$.

z(mm)	σ_x(mm)	σ_y(mm)	z(mm)	σ_x(mm)	σ_y(mm)
2	1,9630	1,8289	19	1,4961	1,3833
3	2,0047	1,8696	20	1,5504	1,4907
4	1,8896	1,7048	21	1,8023	1,8042
5	1,8776	1,6631	22	1,4611	1,5655
6	1,8701	1,6244	23	1,4921	1,6523
7	1,8951	1,6416	24	1,4539	1,6956
8	1,8834	1,6069	25	1,5840	1,8561
9	1,7695	1,4237	26	1,6097	1,9076
10	1,7743	1,4601	27	1,5102	1,9062
11	1,7297	1,3693	28	1,5537	1,9557
12	1,8834	1,6211	29	1,6043	2,0197
13	1,7771	1,4301	30	1,8795	2,1658
14	1,7217	1,3659	35	1,9877	2,1959
15	1,6161	1,2926	40	2,2254	2,2578
16	1,6514	1,3714	45	2,2424	2,3009
17	1,5665	1,3286	50	2,2904	2,3932
18	1,5665	1,4034			

Data analysis: *Gnuplot 3.7* log

```
FIT:    data read from 'M_Sq_47um_x.txt'
        #datapoints = 33

function used for fitting: y(x)
fitted parameters initialized with current variable values

initial set of free parameter values

a               = 2.59015
b               = -0.0399128
c               = 0.00071275

After 5 iterations the fit converged.
final sum of squares of residuals : 0.304469
rel. change during last iteration : -5.03831e-007


Final set of parameters            Asymptotic Standard Error
=======================            ==========================
```

```
a               = 2.58548          +/- 0.05156      (1.994%)
b               = -0.0380862       +/- 0.004939     (12.97%)
c               = 0.000671225      +/- 0.0001007    (15%)


FIT:    data read from 'M_Sq_47um_y.txt'
        #datapoints = 33


function used for fitting: y(x)
fitted parameters initialized with current variable values

initial set of free parameter values

a               = 2.58548
b               = -0.0380862
c               = 0.000671896

After 6 iterations the fit converged.
final sum of squares of residuals : 0.299848
rel. change during last iteration : -1.13831e-009

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

a               = 2.15273          +/- 0.05117      (2.377%)
b               = -0.0249843       +/- 0.004901     (19.62%)
c               = 0.000671818      +/- 9.99e-005    (14.87%)


FIT:    data read from 'M_Sq_52um_x.txt'
        #datapoints = 33


function used for fitting: y(x)
fitted parameters initialized with current variable values

initial set of free parameter values

a               = 2.15273
b               = -0.0249843
c               = 0.00067249

After 6 iterations the fit converged.

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

a               = 2.11433          +/- 0.05832      (2.758%)
b               = -0.0472125       +/- 0.005586     (11.83%)
c               = 0.00109157       +/- 0.0001139    (10.43%)


FIT:    data read from 'M_Sq_52um_y.txt'
        #datapoints = 33
```

```
function used for fitting: y(x)
fitted parameters initialized with current variable values

initial set of free parameter values

a               = 2.11433
b               = -0.0472125
c               = 0.00109266

After 6 iterations the fit converged.

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

a               = 1.64453          +/- 0.09428       (5.733%)
b               = -0.0154143       +/- 0.00903       (58.58%)
c               = 0.000717644      +/- 0.0001841     (25.65%)
```

With this fit values, using formulas in (5), (6) and (7) we finally obtain the M squared factor for each axis, and allows us, at the same time, to plot the graphs below:

$\boldsymbol{\lambda = 47\ \mu m}$:

$$z_{0x} = 28{,}37\ mm;\ \sigma_{0x} = 1{,}43\ mm;$$

$$\boldsymbol{M_x^2 = 9{,}9}$$

$$z_{0y} = 18{,}59\ mm;\ \sigma_{0y} = 1{,}39\ mm;$$

$$\boldsymbol{M_y^2 = 9{,}6}$$

$\boldsymbol{\lambda = 52\ \mu m}$:

$$z_{0x} = 21{,}62\ mm;\ \sigma_{0x} = 1{,}27\ mm;$$

$$\boldsymbol{M_x^2 = 10{,}11}$$

$$z_{0y} = 10{,}73\ mm;\ \sigma_{0y} = 1{,}25\ mm;$$

$$\boldsymbol{M_y^2 = 8{,}1}$$

Conclusions

Since we are expecting values around 2 up to 5 for the M squared factor, we can't say our accuracy has been the better. Might be due to data analysis errors as well as systematic experimental deviation of the measurements.

Figure 1: Quadratic function fitting experimental values of σ_x ($\lambda = 47\ \mu m$).

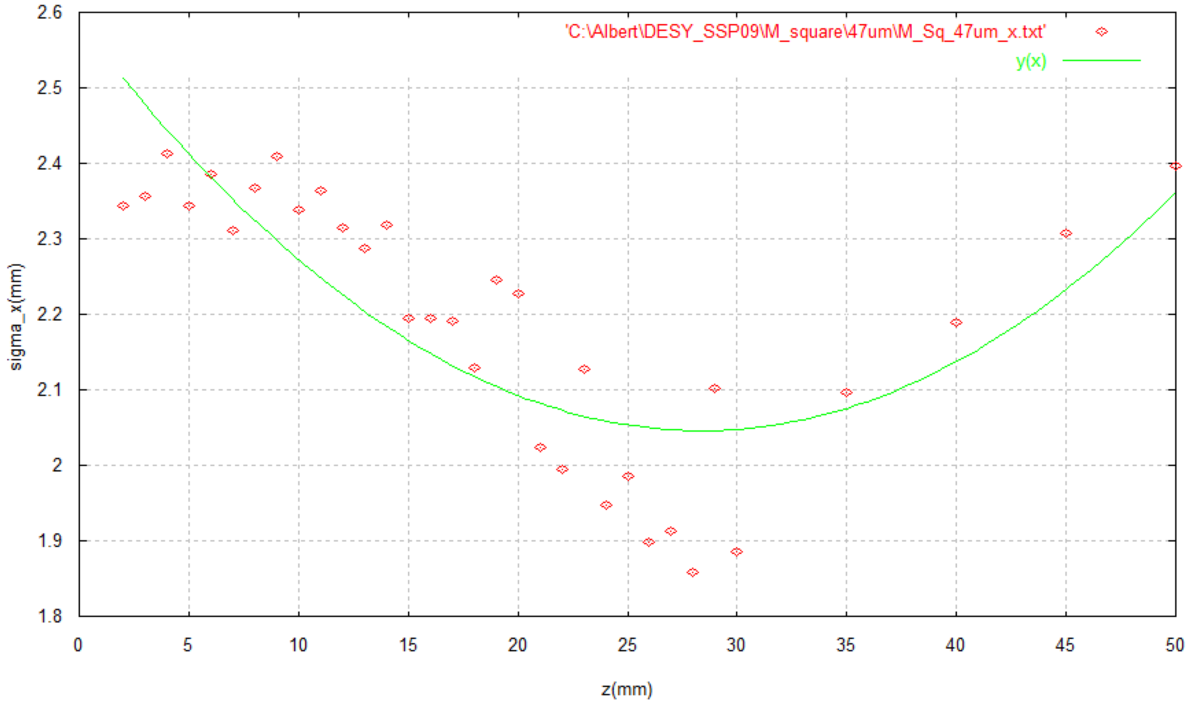


Figure 2: Quadratic function fitting experimental values of σ_y ($\lambda = 47\ \mu m$).

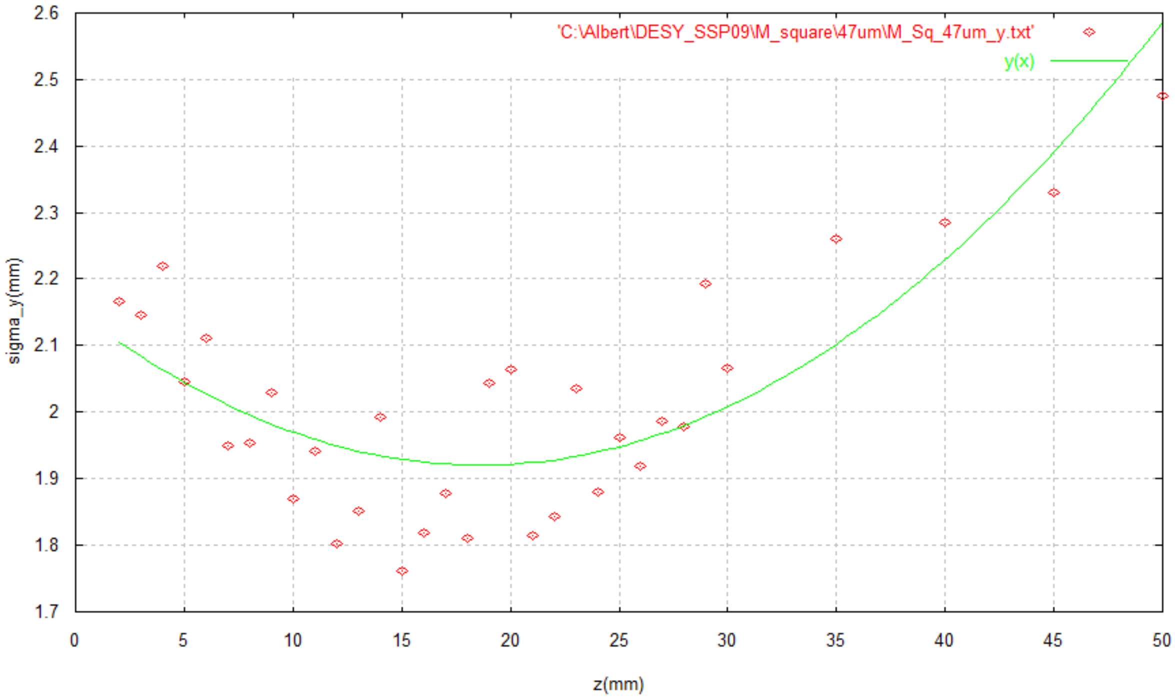


Figure 3: Quadratic function fitting experimental values of σ_x ($\lambda = 52\ \mu m$).

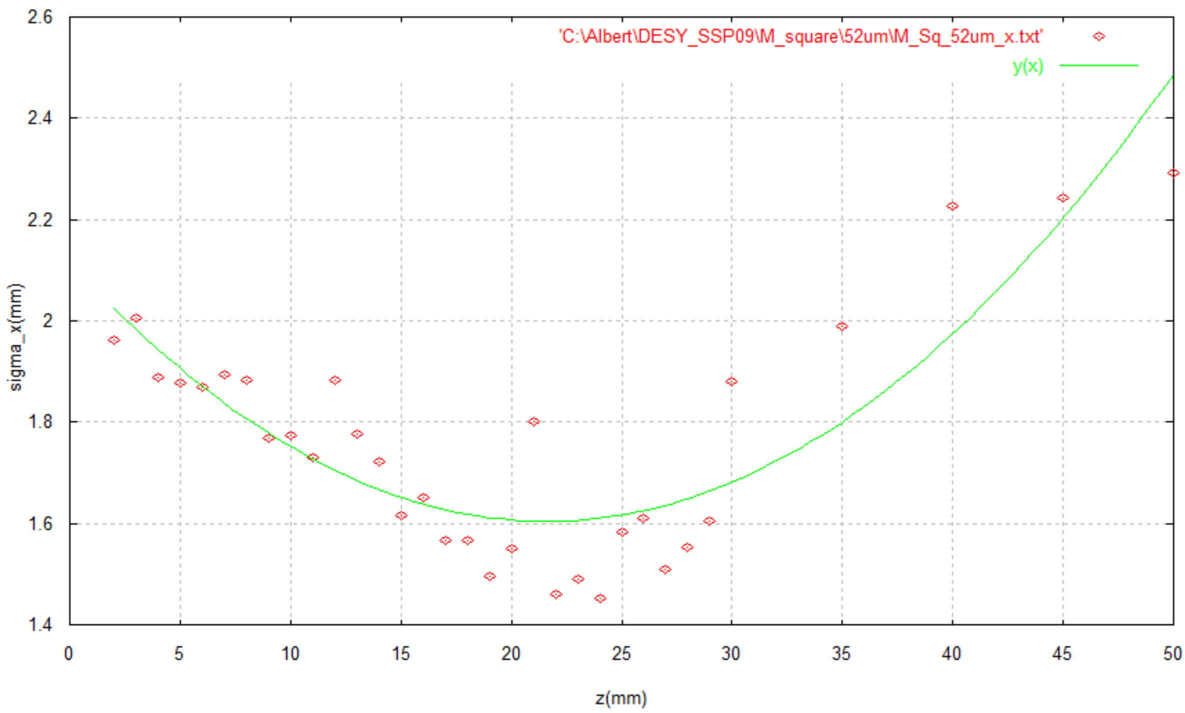
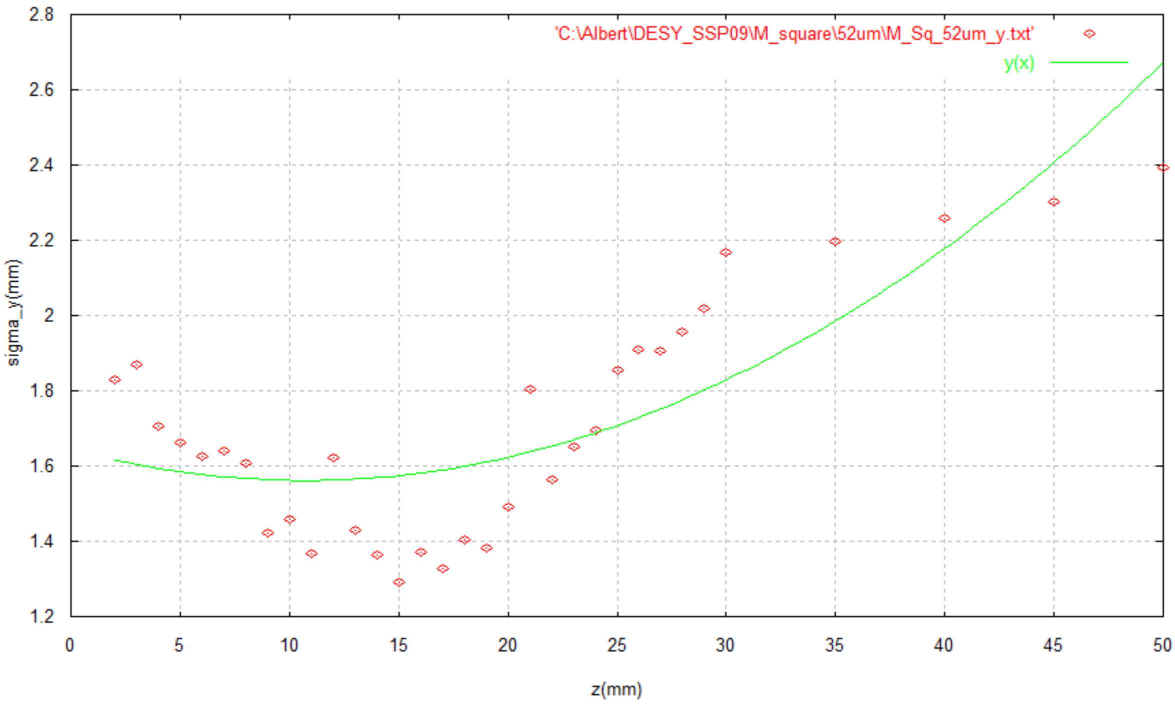


Figure 4: Quadratic function fitting experimental values of σ_y ($\lambda = 52\ \mu m$).



References

http://en.wikipedia.org/wiki/Beam_parameter_product

http://www.stanford.edu/~siegman/beam_quality_tutorial_osa.pdf